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ON INTEGRAL INEQUALITIES OF THE THEORY OF ELASTICO-PLASTIC BODY*

I. A. BEREZHNOI and D. D. IVLEV

Integral inequalities which lead to the Onsager principle of maximum dissipation rate are determined, and the limits imposed by these on the mechanical behavior of material are established.

The relation between integral inequalities of the theory of plasticity which lead to the Mises principle were considered in /1/. The equivalence of the Drucker /2/ and Hill /3/ inequalities, as well as that of Il'iushin /4/ and the one derived in /1/, is demonstrated below.

The problem of constructing the theory of plasticity on the basis of the Onsager principle of maximum dissipation rate was considered, for instance, in /5,6/.

1. Let us consider a work hardening elastico-plastic body. We denote by σ and e the tensors of actual stresses and strains, and assume that

$$e = e^e + e^p \tag{1.1}$$

where e^e and e^p are, respectively, the elastic and plastic components of strain. We assume that the elastic properties of the material are independent of its plastic properties.

Let us consider loading cycles that are closed with respect to stresses and strains. Let BAA_1AB (Fig.1) be the cycle closed with respect to stresses. Let BAB run through the region of elastic deformations, the material reach its elasticity limit at point A, and vector σ emerge at the loading surface. Plastic deformations obtain on the loading segment AA_1 along which we denote the stress increments by $\delta\sigma$ and the corresponding strain increments by δe^p.

In the stress space the open cycle BAA_1AC (Fig.2) corresponds to the stress space to the cycle closed with respect to actual deformations. Total strains at points B and C are by definition equal. We denote by subscripts e and σ the integrals over the cycle closed with respect to strains and stresses, respectively. By definition

$$\oint_e C\, de = 0, \quad C = \mathrm{const} \tag{1.2}$$

Since along segment AA_1 (Fig.2) the actual strains δe^p increase, hence along segment BC the increment Δe^e of elastic strains compensating δe^p must appear, i.e.

$$\delta e^p + \Delta e^e = 0, \quad \Delta e^e = -\delta e^p$$

The stress increment $\Delta\sigma = \sigma_C - \sigma_B$ corresponds (Fig.2) to the increment Δe^e. It is obvious that generally $\delta\sigma \neq \Delta\sigma$.

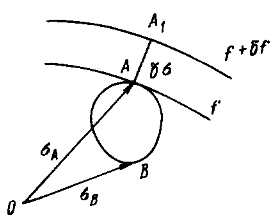

Fig.1

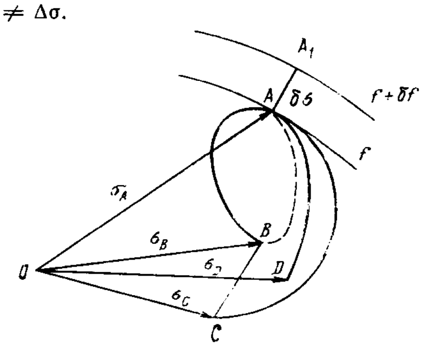

Fig.2

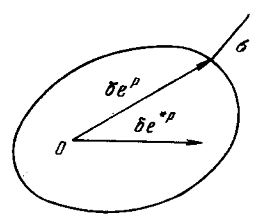

Fig.3

The postulates /1—4/ that form the basis of the Mises principle are of the form

$$\oint_\sigma (\sigma - \sigma_B)\, de \geqslant 0, \quad \oint_e \sigma\, de \geqslant 0, \quad \oint_\sigma e\, d\sigma \leqslant 0, \quad \oint_e (e - e_B{}^e)\, d\sigma \leqslant 0 \tag{1.3}$$

of which the first was proposed in /2/, the second in /3/, the third in /4/, and the fourth in /1/.

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Let us prove the existence of relationships

$$\oint_\sigma (\sigma - \sigma_B)\, de + \oint_\sigma e\, d\sigma \equiv 0, \quad \oint_e \sigma\, de + \oint_e (e - e_B{}^e)\, d\sigma \equiv 0 \tag{1.4}$$

Beginning with the first one, we have

$$\oint_\sigma e\, d\sigma = \oint_\sigma d(\sigma e) - \oint_\sigma \sigma\, de \tag{1.5}$$

Evidently

$$\oint_\sigma d(\sigma e) = \sigma_B \delta e^p = \oint_\sigma \sigma_B\, de \tag{1.6}$$

From (1.5) and (1.6) we obtain

$$\oint_\sigma e\, d\sigma = -\oint_\sigma (\sigma - \sigma_B)\, d\sigma$$

Thus the Drucker and Hill postulates (the first and third of inequalities (1.3)) are equivalent. Let us prove the validity of the second of formulas (1.4). We have

$$\oint_e (e - e_B{}^e)\, d\sigma = \oint_e d(\sigma e) - \oint_e \sigma\, de - e_B{}^e \oint_e d\sigma$$

Taking into account that $e_C = e_B{}^e$, we obtain

$$\oint_e d(\sigma e) = \sigma_C e_C - \sigma_B e_B{}^e = e_B{}^e \Delta\sigma = e_B{}^e \oint_e d\sigma \tag{1.7}$$

From this follows

$$\oint_e (e_B - e_B{}^e)\, d\sigma = -\oint_e \sigma\, de$$

Thus the postulates of Il'iushin and the postulate stated in /2/ (the second and fourth of inequalities (1.3)) are equivalent. It is, therefore, possible to say that there are two independent postulates, viz. Drucker's and Il'iushin's. Hill's postulate and the one formulated in /1/ are different forms of the postulates of Drucker and Il'iushin, respectively.

2. Let us consider the integral inequalities that lead to the Onsager maximum principle. We assume the existence of the dissipative function

$$\sigma e^p = D(\varepsilon^p, e, \chi) \tag{2.1}$$

where $\varepsilon = de/dt = \varepsilon^e + \varepsilon^p$ is the rate of strain and χ are some work hardening parameters. We assume that the dissipative function (2.1) is homogeneous and of first order with respect to the component ε^p.

We introduce the Onsager principle of maximum dissipation in the form proposed by Ziegler /1/

$$(\varepsilon^p - \varepsilon^{*p})\, \sigma \geqslant 0 \tag{2.2}$$

where ε^{*p} is the possible rate of plastic deformation permitted by the specified dissipative function

$$D(\varepsilon^p, e, \chi) \geqslant D(\varepsilon^{*p}, e, \chi) \tag{2.3}$$

From (2.1)—(2.3) follows the associated law of loading

$$\sigma = \partial D / \partial \varepsilon^p \tag{2.4}$$

Formulas (2.1) and (2.4) completely determine the properties of a plastic body. As shown in /6/, (2.1) and (2.4) imply the existence of the function of loading and of the associated flow law

$$f(\sigma, e^p, \chi) = 0, \ \varepsilon^p = \lambda (\partial f / \partial \sigma), \ \lambda \geqslant 0 \tag{2.5}$$

For simplicity, we shall consider the case of smooth loading functions; extension of results to piecewise smooth functions does not present any fundamental difficulties.

Let us consider any possible increment δe^{*p} at point A (Fig.1). Since the dissipative function is homogeneous with respect to ε^p, the inequality (2.3) may be represented in the form

$$D(\delta e^p, e, \chi) \geqslant D(\delta e^{*p}, e, \chi) \tag{2.6}$$

A certain level of the dissipative function $D(\delta e^p, e, \chi) = \text{const}$ which corresponds to actual plastic deformation increment δe^p is shown in Fig.3, together with possible plastic deformation increments δe^{*p}.

Let us assume that segment AA_1 (Fig.1) is fairly small and restrict the analysis to quantities of the first order of smallness. We have

$$\oint_\sigma \sigma\, de = \sigma_A \delta e^p \tag{2.7}$$

Actually, the work of stresses over elastic deformations over the closed cycle of stresses is zero, and the plastic deformation increment is nonzero only at point A.

By definition

$$e^* = e^e + e^{*p}, \quad de^* = de^e + de^{*p} \tag{2.8}$$

where the elastic part of deformation is related to stresses by Hooke's law. The increment δe^{*p} can have any value within the constraint imposed by (2.6). For the considered loading path (Fig.1) we obtain similarly to (2.7)

$$\oint_\sigma \sigma\, de^* = \sigma_A \delta e^{*p} \tag{2.9}$$

Subtracting expression (2.9) from (2.7) we obtain

$$\oint_\sigma \sigma\, (de - de^*) = \sigma_A (\delta e^p - \delta e^{*p}) \tag{2.10}$$

Postulating the inequality

$$\oint_\sigma \sigma\, (de - de^*) \geqslant 0 \tag{2.11}$$

we obtain in conformity with (2.10), as a corollary, the Onsager principle accurate apart from the notation.

In conformity with (1.5), (1.6), and (2.7) we have

$$\oint_\sigma e\, d\sigma = (\sigma_B - \sigma_A)\, \delta e^p \tag{2.12}$$

In the course of its passage over the stress cycle the component e^{*p} acquaires any increments δe^{*p} that satisy the constraint (2.6). Similarly to (2.12) we have

$$\oint_\sigma e^*\, d\sigma = (\sigma_B - \sigma_A)\, \delta e^{*p} \tag{2.13}$$

Substracting (2.13) from (2.12) we obtain

$$\oint_\sigma (e - e^*)\, d\sigma = (\sigma_B - \sigma_A)(\delta e^p - \delta e^{*p}) \tag{2.14}$$

If we set $\sigma_B = 0$, i.e. introducing the loading not at some initial state of stress under the loading surface, but at the zero state of stress from (2.13) we then obtain

$$\oint_\sigma (e - e^*)\, d\sigma = -\sigma_A (\delta e^p - \delta e^{*p}) \tag{2.15}$$

Stipulating the inequality

$$\oint_\sigma (e - e^*)\, d\sigma \leqslant 0 \tag{2.16}$$

and bearing in mind that the loading cycle begins and ends in the unstressed state, we obtain, as a corollary, the Onsager principle.

Let us consider cycles that are closed with respect to deformations. We shall investigate, besides the cycle closed with respect to actual deformations, the cycle closed with respect to possible deformations e^*. In Fig.2 such a cycle is represented by BAA_1ABD on whose segment BA plastic deformations do not develop and, consequently, the properties of this segment are arbitrary. Some plastic deformation increments δe^{*p} correspond to the loading on segment AA_1, and it is then necessary to move to point D at which the increment of elastic deformations compensates increment Δe^{*p}. We denote the respective increment of elastic deformations by Δe_1^e which obviously corresponds to the stress increment $\Delta\sigma_1 = \sigma_D - \sigma_B$. Thus

$$\delta e^{*p} + \Delta e_1^e = 0, \quad \Delta e_1^e = -\delta e^{*p} \tag{2.17}$$

Integration over the closed cycle with respect to possible deformations will be denoted by the subscript e^*. Since no plastic deformations appear under the loading surface, hence it is possible to assume without loss of generality that the paths AC and AD pass through point

B. Then

$$\oint_e = \oint_\sigma + \int_B^C, \quad \oint_{e^*} = \oint_\sigma + \int_B^D \tag{2.18}$$

Let us consider the integral

$$\oint_e \sigma\, de = \oint_\sigma \sigma\, de + \int_B^C \sigma\, de \tag{2.19}$$

Elastic deformation occurs on section BC. Assuming the quantities $\delta e^p = -\Delta e^e$ to be fairly small, we obtain

$$\int_B^C \sigma\, de = \sigma_B \Delta e^e = -\sigma_B \delta e^p \tag{2.20}$$

which is accurate to first order of smallness.

From (2.19), in conformity with (2.7) and (2.20), we have

$$\oint_e \sigma\, de = (\sigma_A - \sigma_B)\, \delta e^p \tag{2.21}$$

Similarly we have

$$\oint_{e^*} \sigma\, de^* = (\sigma_A - \sigma_B)\, \delta e^{*p} \tag{2.22}$$

Subtracting (2.22) from (2.21) we obtain

$$\oint_e \sigma\, de - \oint_{e^*} \sigma\, de^* = (\sigma_A - \sigma_B)(\delta e^p - \delta e^{*p}) \tag{2.23}$$

If $\sigma_B = 0$, then in conformity with (2.23) the inequality

$$\oint_e \sigma\, de - \oint_{e^*} \sigma\, de^* \geqslant 0 \tag{2.24}$$

yields the Onsager principle (2.2).

Thus, when the loading cycle beings at the initial unstressed state, the integral inequality (2.24) yields, as a corollary, the Onsager principle. Let us now consider the integral

$$\oint_e e\, d\sigma = \oint_e d(\sigma e) - \oint_e \sigma\, de \tag{2.25}$$

Using the reciprocity theorem, in accordance with (1.7), we obtain

$$\oint_e d(\sigma e) = e_B{}^e \Delta\sigma = \sigma_B \Delta e^e = -\sigma_B \delta e^p \tag{2.26}$$

In conformity with (2.26) and (2.21) we reduce (2.25) to the form

$$\oint_e e\, d\sigma = -\sigma_A \delta e^p \tag{2.27}$$

and in a similar manner obtain

$$\oint_{e^*} e^*\, d\sigma = -\sigma_A \delta e^{*p} \tag{2.28}$$

Subtracting (2.28) from (2.27) we obtain

$$\oint_e e\, d\sigma - \oint_{e^*} e^*\, d\sigma = -\sigma_A (\delta e^p - \delta e^{*p}) \tag{2.29}$$

By postulating the inequality

$$\oint_e e\, d\sigma - \oint_{e^*} e^*\, d\sigma \leqslant 0 \tag{2.30}$$

we obtain, as a corollary, the Onsager principle for any loading cycles that are closed with respect to strains. Let us show that (2.16) and (2.24) are particular cases of (2.11) and (2.30), respectively. Indeed

$$\oint_\sigma \sigma (de - de^*) + \oint_\sigma (e - e^*)\, d\sigma = \oint_\sigma d(\sigma e) - \oint_\sigma d(\sigma e^*) \tag{2.31}$$

$$\oint_e \sigma\, de - \oint_{e^*} \sigma\, de^* + \oint_e e\, d\sigma - \oint_{e^*} e^*\, d\sigma = \oint_e d(\sigma e) - \oint_{e^*} d(\sigma e^*)$$

Since inequalities (2.16) and (2.24) yield the Onsager principle only for loading cycles beginning at the initially unstressed state, formulas (2.31) are to be used for similar cycles. In that case the loading and deformation cycles also end in zero, and relations (2.31) are identically zero.

Thus the inequalities (2.16) and (2.24) represent other forms of postulates (2.11) and (2.30) for loading cycles beginning at the initially unstressed state. The two postulates (2.11) and (2.30) which yield the Onsager principle when the loading cycles being at any stressed state under the loading surface, are independent. Let us present a symmetric summary of results. The Mises principle

$$(\sigma - \sigma_B)\,\varepsilon^p \geqslant 0$$

is obtained from postulates

$$\oint_\sigma (\sigma - \sigma_B)\,de \geqslant 0, \quad \oint_e (e - e_B{}^e)\,d\sigma \leqslant 0$$

and the Onsager principle

$$(\varepsilon^p - \varepsilon^{*p})\,\sigma \geqslant 0$$

is derived from postulates

$$\oint_\sigma \sigma\,(de - de^*) \geqslant 0, \quad \oint_e e\,d\sigma - \oint_{e^*} e^*\,d\sigma \leqslant 0$$

Note that in our analysis we used only stresses σ and strains e. The integrands are of the form σde and $ed\sigma$, integration is carried out over closed stress and strain cycles, and all four possible combinations of integrals and integrands are investigated. The mechanism of plasticity and elasticity is not specified, except that e^e and e^p are, respectively, the reversible and residual deformations. Specific determination of stresses and strains is not used; it is necessary that condition (2.1) is satisfied besides (1.1).

3. Let us consider the basic inequalities of the theory of plasticity with an accuracy up to second order of smallness.

The Drucker inequality (the first of inequalities (1.3)) is obviously of the form

$$\oint_\sigma (\sigma - \sigma_B)\,de = \left(\sigma_A - \sigma_B + \frac{1}{2}\delta\sigma\right)\delta e^p \geqslant 0 \qquad (3.1)$$

Let us consider the Il'iushin inequality (the second of (1.3))

$$\oint_e \sigma\,de = \oint_\sigma \sigma\,de + \oint_B^C \sigma\,de = \left(\sigma_A + \frac{1}{2}\delta\sigma\right)\delta e^p + \left(\sigma_B + \frac{1}{2}\Delta\sigma\right)\Delta e^e = (\sigma_A - \sigma_B)\,\delta e^p + \frac{1}{2}(\delta\sigma\delta e^p + \Delta\sigma\Delta e^e) \geqslant 0 \qquad (3.2)$$

If $A = B$ (Fig.4), (3.1) and (3.2) reduce, respectively, to the inequalities

$$\delta\sigma\delta e^p \geqslant 0 \qquad (3.3)$$

$$\delta\sigma\delta e^p + \Delta\sigma\Delta e^e \geqslant 0, \; \delta e^p + \Delta e^e = 0 \qquad (3.4)$$

The Hill inequality (the third of (1.3)) obviously reduces to (3.3), and the fourth of inequalities (1.3) to (3.4).

A material that satisfies condition (3.3) is usually called stable. The dependence $\sigma - e$ for uniaxial tension is shown in Fig.5, where point A corresponds to the yield stress.

Since in uniaxial tension $\delta e^p > 0$, hence in conformity with (3.3) $\delta\sigma \geqslant 0$. The relation $\sigma - e$ /between σ and e / is represented in that case by a monotonically increasing curve (AB in Fig.5).

We define the dependence $\delta\sigma - \delta e^p$ (relation between $\delta\sigma$ and δe^p) by

$$\delta\sigma = E_s\delta e^p \qquad (3.5)$$

where E_s is the plasticity modulus.

Condition (3.4) for uniaxial tension is of the form

$$\delta\sigma \geqslant \Delta\sigma = E\Delta e^e = -E\delta e^p \qquad (3.6)$$

where E is the elasticity modulus for such tension. From (3.5) and (3.6) we have $E_s \geqslant -E$. Consequently, the curve of $\sigma - e$ has in this case a monotonically decreasing section (AB_1 in Fig.5), which means the behavior of the material may be unstable. If we set $E = \mathrm{tg}\alpha$, then in conformity with (3.6) the slope of /the curve representing/ the section of unstable behavior of the material cannot be less than $-\alpha$ (Fig.5). Condition (3.4) evidently comprises condition (3.3) and, thus, defines a wider class of materials. For instance, the behavior of a material in the plastic region may be defined by the curve AB_2 in Fig.5. Nevertheless the Mises principle and consequently, the associated law of plastic flow are valid for that wider class of materials.

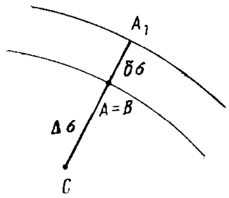

Fig.4

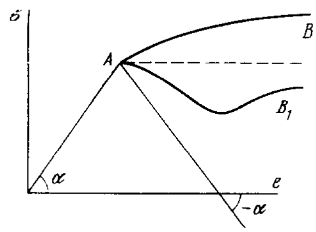

Fig.5

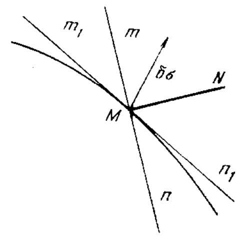

Fig.6

Let us consider the corollaries of inequalities (3.3) and (3.4) in the general case of stressed state. Note that (3.3) does not directly imply either the nonconcavity of the loading surface or the associated flow law. It is possible to construct a model of a plastic body whose properties conform to Fig.6. The plastic strain increment is nonorthogonal to the loading surface and directed along segment MN. The straight line mn is orthogonal to MN and the straight line m_1n_1 is tangent to the loading surface at point M. According to (3.3) the plastic strain increment occurs when vector $\delta\sigma$ issuing from point M lies within sectors mMN or nMN. A stress increment within sector mMm_1 does not produce a plastic strain increase, but a displacement of the loading surface occurs in the neighborhood of point M; there is only a shift of the yield stress. Increment $\delta\sigma$ within segment nMn_1 corresponds to unstable plastic deformation. The loading surface shifts in the neighborhood of the point M inward of the loading surface previous state.

Such a model is entirely admissible when only inequality (3.3) is taken as the basis. Its validity is vitiated by corollaries of the Mises principle.

The associated flow law can be obtained from (3.3) if one assumes that the increment $\delta\sigma$ which produces plastic deformation cannot be directed inward from the loading surface. This is equivalent to the assumption that the material is stable.

Condition (3.3) is actually the condition of stability of the material if it conforms to the Mises (or Onsager) postulate whose corollaries are the nonconcavity of the loading surface and the associated flow law. Condition (3.3), as well as (3.4), determines the possible directions of the increment $\delta\sigma$ which produces plastic deformation increments. Condition (3.4) admits increments of plastic deformations for inward motion from the loading surface.

We formulate the generalized inequality as

$$(1-a)\oint_{\sigma}(\sigma-\sigma_B)\,de - a\oint_{e}(e-e_B{}^{e})\,d\sigma \geqslant 0, \quad a=\text{const} \tag{3.7}$$

The equivalent inequality is

$$a\oint_{e}\sigma\,de-(1-a)\oint_{\sigma}e\,d\sigma\geqslant 0, \quad a=\text{const} \tag{3.8}$$

When $a=0$, formulas (3.7) and (3.8) evidently reduce to the first and the third of inequalities (1.3), respectively. When $a=1$ formulas (3.7) and (3.8) reduce to the fourth and second of inequalities (1.3). In conformity with (3.1), (3.2) and (1.4), formulas (3.7) and (3.8) are, within higher order smallness, of the form

$$(\sigma_A-\sigma_B)\,\delta e^p+\frac{1}{2}(\delta\sigma\delta e^p+a\Delta\sigma\Delta e^e)\geqslant 0 \tag{3.9}$$

Setting $A=B$ (Fig.4) we obtain from (3.9) the generalized condition of plastic loading

$$\delta\sigma\delta e^e+a\Delta\sigma\Delta e^p\geqslant 0, \quad \delta e^p+\Delta e^e=0 \tag{3.10}$$

If $a>0$, $\delta\sigma\delta e^p$ may become negative, plastic deformations can appear when $\delta\sigma$ is directed inward from the loading surface, and the behavior of the material can be unstable.

If $a<0$, $\delta\sigma\delta e^p$ may assume limited positive values. For uniaxial tension from (3.10) we obtain

$$\delta\sigma\geqslant E_s{}^{\circ}\delta e^p, \quad E_s{}^{\circ}=-aE, \quad a<0 \tag{3.11}$$

In conformity with (3.11) and (3.5), the modulus E_s cannot be lower than the entirely determinate quantity $E_s{}^{\circ}$.

Let us consider inequalities (2.11) and (2.32) within quantities of second order of smallness. Similarly to (3.1) and (3.2) we obtain

$$\oint_{\sigma}\sigma(de-de^*)=\sigma_A(\delta e^p-\delta e^{*p})+\frac{1}{2}\delta\sigma(\delta e^p-\delta e^{*p})\geqslant 0 \tag{3.12}$$

$$\oint_{e}e\,d\sigma-\oint_{e^*}e^*\,d\sigma=\sigma_A(\delta e^p-\delta e^{*p})+\frac{1}{2}(\delta\sigma\delta e^p+\Delta\sigma\Delta e^e-\Delta\sigma\delta e^{*p}-\Delta\sigma_1\Delta e_1{}^e)\leqslant 0$$

Unlike in the previously considered cases, it is not possible to get rid of the first term in the right-hand sides of both equalities (3.12), and the second terms of these do not essentially affect inequalities (3.12).

Let us make a few remarks on the principle of maximum of dissipation rate. Onsager had formulated the principle /5/ according to which actual deformation rates ε^p maximize the expression in our notation

$$\sigma\varepsilon^{*p} - \frac{1}{2} D(\varepsilon^p) = \sigma\cdot(\varepsilon^{*p} - \frac{1}{2}\varepsilon^p) \tag{3.13}$$

Substituting in (3.13) the quantity ε^p for ε^{*p} we find that the expression $\sigma\varepsilon^p = D(\varepsilon^p)$ for the dissipation rate is maximized.

The Onsager principle of maximum dissipation rate can, obviously, be formulated as follows: actual deformation rates maximize the expression

$$\sigma(\varepsilon^{*p} - m\varepsilon^p), \quad 0 \leqslant m < 1 \tag{3.14}$$

The alternative expression for $m = 1$ formulated by Ziegler /5/ is of the form

$$\sigma(\varepsilon^p - \varepsilon^{*p}) \geqslant 0 \tag{3.15}$$

Consequently the Onsager principle of maximum dissipation rate can be expressed in the form (3.14) or (3.15).

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